Efficient Multiplication of Somewhat Small Integers using Number-Theoretic Transforms

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From integers to polynomials and back

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(The reverse reduction works as well, using *Kronecker substitution*: Given $f, g \in \mathbb{Z}[x]$, choose large 2^ℓ, compute $c = f(2^\ell) \cdot g(2^\ell)$, and recover $f \cdot g$ from c via ℓ-bit chunking.)

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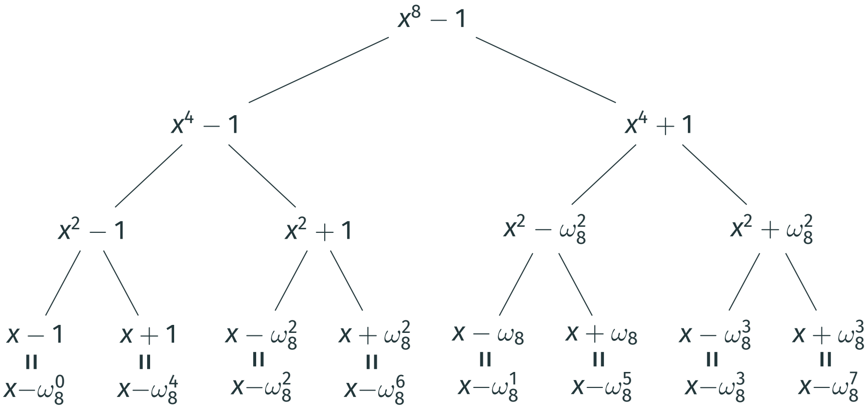
$\implies$ [FFT + pointwise multiplication + inverse FFT] is only $O(n \log n)$ operations in R.

FFT tree for $R[x]/(x^{2^m}-1)$



Computing the FFT (and inverse FFT)

Going <u>down</u> one layer: Compute $R[x]/(x^{2k} - \tau^2) \xrightarrow{\sim} R[x]/(x^k - \tau) \times R[x]/(x^k + \tau)$.

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$\implies$ Work per layer is $O(n)$, and there are $O(\log n)$ layers. $\implies O(n \log n)$.

Butterflies

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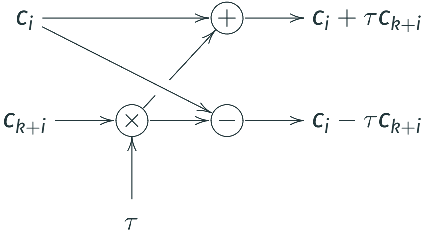
Cooley–Tukey butterfly



Reduce $f = c_0 + c_1 x + \ldots + c_{2k-1} c^{2k-1}$ modulo $(x^k - \tau)$ and $(x^k + \tau)$.

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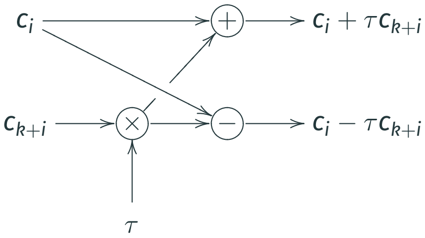

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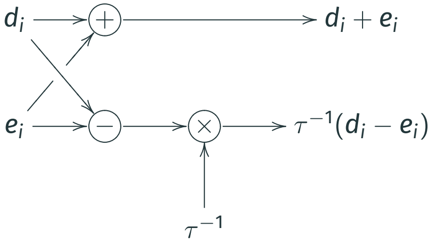

Recover $2 \cdot f \in R[x]/(x^{2k} - \tau^2)$
from $\left(f \bmod (x^k - \tau), f \bmod (x^k + \tau)\right)$.

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- Compute NTT modulo q_1 and q_2 separately, recombine via CRT $\mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \xrightarrow{\sim} \mathbb{Z}/q$.

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Our algorithm isn't even properly specified for arbitrary lengths. If it were, it would scale worse than Schönhage–Strassen. Still, it appears to be *faster for medium-sized integers!*

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Compare conventional wisdom:

“[Schönhage–Strassen] starts to outperform [...] for numbers beyond $2^{2^{15}}$ *to* $2^{2^{17}}$*.”*

(Wikipedia)

Target Architectures

- Focus on 32-bit Arm microcontrollers
- First target: Arm Cortex-M3
  - Announced in 2004
  - Implements Armv7-M
  - Interesting/dangerous feature: Timing of long multiplications (e.g., `UMULL`) is input-dependent
    $\rightarrow$ Avoid for constant-time code
  - We make use of STM32 Nucleo-F207ZG with the STM32F207ZG
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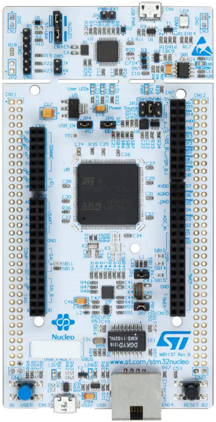

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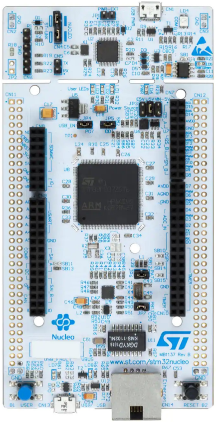

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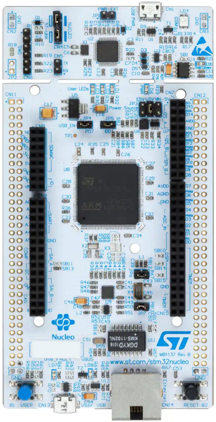

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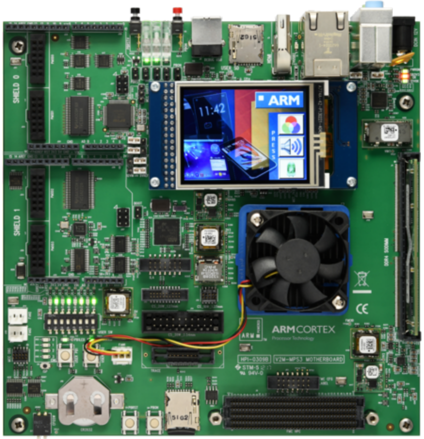

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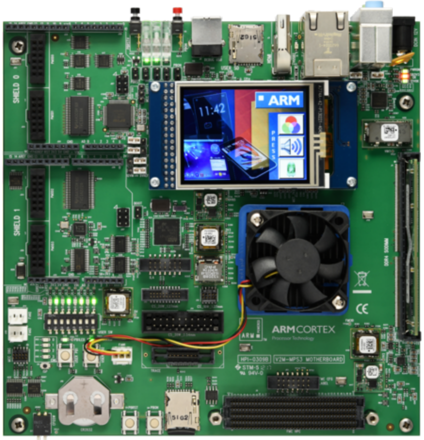

Fermat Number Transforms (FNT)

- Recall: For NTTs we require $2^k \mid q - 1$ with prime q
- Fermat numbers: $2^{2^k} + 1$
- Fermat primes: 3, 5, 17, 257, 65537
- Example: 65537
  - $\omega_2 = -1 = 2^{16}$
  - $\omega_4 = 2^8$
  - $\omega_8 = 2^4$
  - $\omega_{16} = 2^2$
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Parameter Choices

- High-level goal: Efficient N-bit (2048, 4096) multiplication
  - Chunk up number in ℓ-bit coefficients
  - Pad with zeros to have an n-coefficient polynomial
  - Each coefficient is modulo small $q = q_1 q_2$
  - Perform a k-layer NTT-based multiplication
- Constraint 1: We want to make efficient use of the available multipliers
  - M3:
    `mul` $32 \times 32 \rightarrow 32$ bit (low multiplication)
    $\rightarrow$ want to limit moduli q_i to 16 bit
    $\rightarrow$ special case: FNT with $q_2 = 65537$ for NTT
  - M55:
    `vmul` $32 \times 32 \rightarrow 32$ bit (low multiplication)
    `vqrdmulh`: $32 \times 32 \rightarrow 32$ bit (rounding doubling high multiplication)
    $\rightarrow$ want to limit moduli q_i to 32 bit

Parameter Choices

- High-level goal: Efficient N-bit (2048, 4096) multiplication
  - Chunk up number in ℓ-bit coefficients
  - Pad with zeros to have an n-coefficient polynomial
  - Each coefficient is modulo small $q = q_1 q_2$
  - Perform a k-layer NTT-based multiplication
- Constraint 1: We want to make efficient use of the available multipliers
  - M3:
    `mul` $32 \times 32 \rightarrow 32$ bit (low multiplication)
    $\rightarrow$ want to limit moduli q_i to 16 bit
    $\rightarrow$ special case: FNT with $q_2 = 65537$ for NTT
  - M55:
    `vmul` $32 \times 32 \rightarrow 32$ bit (low multiplication)
    `vqrdmulh`: $32 \times 32 \rightarrow 32$ bit (rounding doubling high multiplication)
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Parameter Choices (2)

- Constraint 2: Need to be able to represent the $2N$-bit result
  $\rightarrow n \geq \lceil 2N/\ell \rceil$
- Constraint 3: n should be NTT-friendly
  $\rightarrow$ power of two or small multiple of power of two
- Constraint 4: Require NTT-friendly modulus
  $\rightarrow$ restrict to prime q_1, q_2
  $\rightarrow 2^k \mid q_1 - 1$ and $2^k \mid q_2 - 1$ to have the required principal roots of unity
- Constraint 5: Coefficients of polynomial product must not overflow q
  $\rightarrow q \geq \lceil N/\ell \rceil \cdot 2^{2\ell}$

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Parameter Choices (3)

Cortex-M3				
bits (N)	chunking (ℓ)	poly length (n)	NTT	modulus $q = q_1 \cdot q_2$
2048	11 bits	384	$128 = 2^7$	$12289 \cdot 65537$
4096	11 bits	768	$256 = 2^8$	$25601 \cdot 65537$

Cortex-M55				
bits (N)	chunking (ℓ)	poly length (n)	NTT	modulus $q = q_1 \cdot q_2$
2048	22 bits	192	$64 \cdot 3 = 2^6 \cdot 3$	$114\,826\,273 \cdot 128\,919\,937$
4096	22 bits	384	$128 \cdot 3 = 2^7 \cdot 3$	$114\,826\,273 \cdot 128\,919\,937$

Low-level: Modular Coefficient Multiplication on Cortex-M3

NTT: Montgomery mult

`mul a, a, b`

`mul t, a,` $-q^{-1} \bmod {}^{\pm}2^{16}$

`sxth t, t`

`mla a, t, q, a`

`asr a, a, #16`

NTT: Barrett reductions

`mul t, a,` $\lceil R/q \rfloor$

`add t, t, #(R/2)`

`asr t, t,` #$\log_2$ `R`

`mls a, t, q, a`

FNT: Reduction mod 65537

`ubfx t, a, #0, #16`

`sub a, t, a, asr#16`

Low-level: Modular Coefficient Multiplication on Cortex-M55

- We make use of “Barrett multiplication” from Becker–Hwang–Kannwischer–Yang–Yang (CHES 2022) `https://tches.iacr.org/index.php/TCHES/article/view/9295`
- Pre-compute: $b' = \frac{\lfloor b2^{32}/q \rceil}{2}$
- Implement 4 parallel Barrett multiplications

```
vmul l, a, b
vqrdmulh h, a, b'
vmla l, h, q
```

Application: RSA

- Integer multiplication is dominating operation within RSA
- Need to compute `expmod` modulo $n = pq$ (4096-bit n, 2048-bit p, q)
- Encryption:
  $c = m^e \bmod n$ (usually, $e = 65537$)
  $\rightarrow$ requires 4096-bit multiplication; e may leak via timing
- Decryption:
  $c^d \bmod n = CRT(c^d \bmod p, c^d \bmod q)$
  $\rightarrow$ requires 2048-bit multiplication; d must not leak via timing
- Fixed-window exponentiation for decryption
  $\rightarrow$ Use constant-time table look-up!

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RSA: `modmul`

- Within `expmod`, we need a `modmul`
- Common way to implement `modmul`: Montgomery multiplication

  $c = a \cdot b$

  $t = c \cdot p^{-1} \bmod R$

  $r = (c - t \cdot p)/R$
- We can actually implement this using NTTs:

  $c = iNTT(NTT(a) \circ NTT(b))$

  $t = iNTT(NTT(c \bmod R) \circ NTT(p^{-1} \bmod R))$

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- We can pre-compute $NTT(p)$ and $NTT(p^{-1} \bmod R)$
- Need $4\times$ NTT and $3\times$ iNTT
- Squaring: $a = b \rightarrow$ only $3\times$ NTT

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Results: Cortex-M3

	n	`mulmod`	`sqrmod`	$\texttt{expmod}_{\text{public}}$	$\texttt{expmod}_{\text{private}}$
This work	2048	220 047	196 830	4 227 473	494 923 435
This work (FIOS)		234 041	–	4 912 705	543 648 872
BearSSL[1]		283 038	–	18 350 210	718 347 177
This work	4096	510 708	454 128	9 752 690	2 250 748 647
This work (FIOS)		926 523	–	19 458 326	4 228 661 467
BearSSL[1]		1 102 151	–	70 443 207	5 505 856 187

RSA-2048 using CRT for decryption

[1] `https://bearssl.org/`

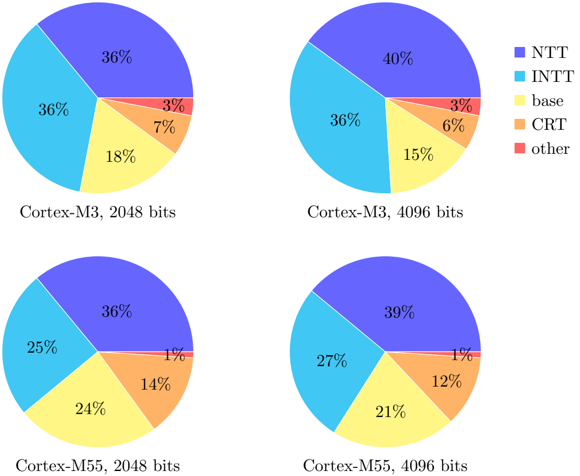
Results: Cortex-M55

	n	mulmod	sqrmod	expmod$_{\text{public}}$	expmod$_{\text{private}}$
This work	2048	21 330	19 701	389 482	50 085 366
This work (FIOS)		20 260	–	426 707	50 683 718
MbedTLS[1]		41 443	–	884 416	108 441 240
BearSSL[2]		83 517	–	5 400 650	217 123 645
This work	4096	47 660	43 620	861 450	218 110 707
This work (FIOS)		73 316	–	1 540 685	358 080 308
MbedTLS[1]		152 371	–	3 223 797	755 391 521
BearSSL[2]		328 801	–	21 254 533	1 646 834 048

RSA-2048 using CRT for decryption

[1] https://github.com/Mbed-TLS/mbedtls
[2] https://bearssl.org/

Profiling of `mulmod`



Conclusions

- NTT-based integer multiplication can be superior for relatively small sizes
  - We implemented 2048-bit and 4096-bit multiplications
  - We target two common Arm platforms: Cortex-M3 and Cortex-M55
- Progress in post-quantum cryptography (lattice-based crypto) helps speeding up pre-quantum crypto
- NTT are much easier to vectorize than other integer-multiplication algorithms
  - Gives advantage on platforms supporting vector instructions, e.g., Cortex-M55

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Conclusions: Limitations

- Limited to certain integer sizes
- Limited to chosen platforms (Cortex-M3, Cortex-M55)
- Our code is heavily unrolled
  - May be problematic on the Cortex-M3 due to ROM/flash constraints
  - Performance overhead of re-rolling the code is hopefully small
- `expmod` allows some pre-computation (modulus and its inverse in NTT domain) favouring NTT-based multiplication
  - General-purpose modular multiplication will be slower

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- How does this translate to other platforms?
  - Unclear
  - For example, Arm Cortex-M4 has powerful multiplication instructions (single cycle `umaal`) that help schoolbook much more than NTTs
  - Armv8-A/Armv9-A processors would be interesting to look at in the future
- Can we scale this to any size of integers (≥ 2048)?
  - Unclear
  - Parameters have been carefully picked for the two sizes (2048, 4096)
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Thanks!

https://eprint.iacr.org/2022/439

https://github.com/ntt-int-mul/ntt-int-mul-m3

https://gitlab.com/arm-research/security/pqmx